

The black-hole area spectrum in the tunneling formalism with GUP

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Abstract

The black hole area spectrum has been studied in the framework of tunneling mechanism, where a Generalized Uncertainty Principle (GUP) has been considered. The results implies in a non evenly spaced spectrum for the black hole area which becomes increasingly spaced as the black hole evaporates.

1. Introduction

The search for a common description of particle physics and gravity and for a quantum theory of the gravitational sector is certainly one of the most outstanding and longstanding problems in physics. Despite important discoveries, at present a reliable theoretical framework is lacking and even the very meaning of quantum spacetime is not clear.

Hawking discovery that black holes have thermodynamics properties like entropy and temperature [1, 2, 3] has established profound links between black hole physics and such seemingly very distant fields as thermodynamics, information theory, and quantum theory, and have opened the doors for discussions that have improved our understanding about possible the features of a further theory of quantum gravity. In this way, the study of black holes has a significance going far beyond astrophysics. Actually, black holes may play a major role in our attempts to shed some light on the nature of a quantum theory of gravity such as the role played by atoms in the early development of quantum mechanics.

A consequence of Hawking calculations is the discovery that, due to the instability of the vacuum in the strong gravitational field of a black hole, these objects are sources of quantum radiation. In the Hawking's original semiclassical approach, the black hole radiation is continuous. However, several authors

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have raised the possibility that Hawking radiation might in fact have a discrete spectrum. In its earliest form, this argument traces to Bekenstein's proposal that the eigenvalues of the black hole event horizon area are of the form [4]

$$A_n = \gamma n l_P^2, \quad (1)$$

where n ranges over positive integers, and $l_P = (G\hbar/c^3)^{1/2}$ is the Planck length.

Diverse computations have been employed in order to find out the correct form of the black hole area spectrum, including some efforts which have been done in order to confirm the equally spaced spectrum suggested by Bekenstein [5, 6, 7, 8, 9, 10]. However, the correct form of the black hole area spectrum remains an open issue.

In this paper we study the black hole area spectrum in the framework of tunneling mechanism, where a Generalized Uncertainty Principle (GUP) has been considered. Among other things, it is a common belief that the Heisenberg uncertainty principle has to be replaced by the so-called generalized uncertainty principle (GUP) when gravitational interactions are taken into account. Banerjee et al [10], by working with tunneling formalism, have employed a calculation which have confirmed the equally spaced spectrum proposed by Bekenstein. However this result depends on the use of the ordinary Heisenberg Uncertainty Principle, in a way that quantum gravity effects have not been appropriately included.

This paper is organized as follows: in section (2), we address the tunneling treatment to black hole evaporation. In section (3), we discuss the Generalized Uncertainty Principle(GUP) from a micro-black hole gedanken experiment. In section (4), we employ the calculation for the black hole area spectrum. The section (5) is devoted to conclusions. In this work we shall consider all fundamental constants (c, \hbar, k_B, G) equal to one.

2. The tunneling process

In the region near the horizon, the effective potential vanishes and there are no grey-body factors. However, the self-consistency of the approach can be seen by recalling that the emission spectrum obtained from these modes is purely thermal. This justifies ignoring the grey-body factors. Moreover, in this region, the theory is dimensionally reduced to a 2-dimensional theory [11, 12] whose metric is just the $(t - r)$ sector of the original metric while the angular part is red-shifted away. Consequently the near-horizon metric has the form

$$ds^2 = F(r)dt^2 + dr^2 F(r). \quad (2)$$

The horizon is defined by the relation $F(r = r_H) = 0$ and the surface gravity is given by $\kappa = \frac{F'(r_H)}{2}$.

The massless Klein-Gordon equation $g^{\mu\nu}\nabla_\mu\nabla_\nu\phi = 0$ under the metric given in (2), is

$$-\frac{1}{F(r)}\partial_t^2\phi + F'(r)\partial_r\phi + F(r)\partial_r^2\phi = 0. \quad (3)$$

Taking the standard WKB ansatz $\phi(r, t) = e^{-\frac{i}{\hbar}S(r, t)}$ and substituting the expansion for $S(r, t)$

$$S(r, t) = S_0(r, t) + \sum_{i=1}^{\infty} \hbar^i S_i(r, t) \quad (4)$$

in (3), we obtain the solutions for ϕ in the semiclassical limit [13, 14]

$$\begin{aligned} \phi_{in}^{(R)} &= e^{-i\omega u_{in}} \quad ; \quad \phi_{in}^{(L)} = e^{-i\omega v_{in}} \\ \phi_{out}^{(R)} &= e^{-i\omega u_{out}} \quad ; \quad \phi_{out}^{(L)} = e^{-i\omega v_{out}} \end{aligned} \quad (5)$$

where the quantity ω is the energy of the particle as measured by an asymptotic observer. Here “ $R(L)$ ” refers to the outgoing (ingoing) mode, while “in(out)” stands for inside (outside) the event horizon. The null coordinates (u, v) are defined as

$$\begin{aligned} u &= tr^*; \\ v &= t + r^*; \\ dr^* &= drF(r). \end{aligned} \quad (6)$$

In the context of the tunneling formalism, a virtual pair of particles is produced in the black hole. One member of this pair can quantum mechanically tunnel through the horizon. This particle is observed at infinity while the other goes towards the center of the black hole. While crossing the horizon the nature of the coordinates changes. This can be accounted by working with Kruskal coordinates which are viable in both sectors of the black-hole event horizon. The Kruskal time (T) and space (X) coordinates inside and outside the horizon are defined as [15]

$$\begin{aligned} T_{in} &= e^{\kappa r_{in}} \cosh(\kappa t_{in}); \quad X_{in} = e^{\kappa r_{in}} \sinh(\kappa t_{in}); \\ T_{out} &= e^{\kappa r_{out}} \sinh(\kappa t_{out}); \quad X_{out} = e^{\kappa r_{out}} \cosh(\kappa t_{out}). \end{aligned} \quad (7)$$

These two sets of coordinates are connected through the following relations:

$$\begin{aligned} t_{in} &= t_{out} - i\frac{\pi}{2\kappa}, \\ r_{in} &= r_{out} + i\frac{\pi}{2\kappa}. \end{aligned} \quad (8)$$

In this way, the Kruskal coordinates get identified as

$$\begin{aligned} T_{in} &= T_{out} , \\ X_{in} &= X_{out} . \end{aligned} \quad (9)$$

Employing equations (8) in equation (6), we can obtain the relations that connect the radial null coordinates defined inside and outside the black-hole event horizon

$$\begin{aligned} u_{in} &= t_{in} - r_{in} = u_{out} - i\pi/\kappa , \\ v_{in} &= t_{in} + r_{in} = v_{out} . \end{aligned} \quad (10)$$

Under these transformations the modes in equations (5) which are traveling in the “in” and “out” sectors of the black-hole horizon are connected through the expressions

$$\begin{aligned} \phi_{in}^{(R)} &= e^{-\frac{\pi\omega}{\kappa}} \phi_{out}^{(R)} , \\ \phi_{in}^{(L)} &= \phi_{out}^{(L)} . \end{aligned} \quad (11)$$

Concentrating on the modes located inside the horizon, the L mode is trapped while the R mode tunnels through the horizon [13, 15]. The probability for the R mode to travel from the inside to the outside of the black hole, as measured by an external observer, is given by

$$P^R = |\phi_{in}^{(R)}|^2 = |e^{-\frac{\pi\omega}{\kappa}} \phi_{out}^{(R)}|^2 = e^{\frac{2\pi\omega}{\kappa}} , \quad (12)$$

where equation (11) has been used to extract the final expression. Since the measurement is done from the outside, $\phi_{in}^{(R)}$ has to be expressed in terms of $\phi_{out}^{(R)}$. Therefore the average value of the energy, measured from outside, is written as

$$\langle \omega \rangle = \frac{\int_0^\infty d\omega \, \omega P^R}{\int_0^\infty d\omega \, P^R} = T_H . \quad (13)$$

where $T_H = \frac{\kappa}{2\pi}$ is the Hawking temperature. In a similar way, one can compute the average squared energy of the particle, detected by an asymptotic observer,

$$\langle \omega^2 \rangle = \frac{\int_0^\infty d\omega \, \omega^2 P^R}{\int_0^\infty d\omega \, P^R} = 2T_H^2 . \quad (14)$$

Hence it is straightforward to evaluate the uncertainty in the detected energy ω by combining equations (13) and (14),

$$(\Delta\omega) = \sqrt{\langle \omega^2 \rangle - \langle \omega \rangle^2} = T_H , \quad (15)$$

which is nothing but the Hawking temperature T_H .

3. The generalized uncertainty principle

The generalized uncertainty principle arises from the Heisenberg uncertainty principle when gravity is taken into account. In this section, we will derive a GUP via a micro black hole gedanken experiment, following closely the content of [16]. When we measure a position with precision of order ΔX , we expect quantum fluctuations of the metric field around the measured position with energy amplitude

$$\Delta\omega \sim \frac{1}{2\Delta X}. \quad (16)$$

The Schwarzschild radius associated with the energy $\Delta\omega$

$$R_S = 2\Delta\omega \quad (17)$$

falls well inside the interval Δx for practical cases. However, if we want to improve the precision indefinitely, the fluctuation $\Delta\omega$ would grow up and the corresponding R_S would become larger and larger, until it reaches the same size as ΔX . As it is well known, the critical length is the Planck length, and the associated energy is the Planck energy ε_P .

If we tried to further decrease ΔX , we should concentrate in that region an energy greater than the Planck energy, and this would enlarge further the Schwarzschild radius R_S , hiding more and more details of the region beyond the event horizon of the micro hole. The situation can be summarized by the inequalities

$$\Delta X = \begin{cases} \frac{1}{2\Delta\omega} & \text{for } \Delta\omega \leq \varepsilon_P, \\ 2\Delta\omega & \text{for } \Delta\omega > \varepsilon_P, \end{cases}$$

which, if combined linearly, yield

$$\Delta X \geq \frac{1}{2\Delta\omega} + 2\Delta\omega. \quad (18)$$

This is a generalization of the uncertainty principle to cases in which gravity is important, i.e. to energies of the order of ε_P . We note that the minimum value of ΔX is reached for $(\Delta\omega)_{min} = \varepsilon_P$ and is given by $(\Delta X)_{min} = 2l_P$.

4. The black hole area spectrum in the tunneling formalism with GUP

In this section, we will derive the black hole area spectrum using the generalized uncertainty process in the quantum tunneling formalism addressed in the section (2).

According to Bekenstein [17], the change in area of a black hole caused either by an absorption or by an emission of a particle is given by

$$\Delta A \geq 8\pi \int_V x T_{00} dV , \quad (19)$$

where x is the distance of the center of mass of the particle from the horizon and T_{00} represents the energy density corresponding to the particle. Here V stands for the volume (a 3-surface) of the system, i.e. the black hole and the particle, outside the black hole, at a constant time.

Following [10], we can consider, based on dimensional grounds, the position x of the emitted particle to be of the order of the uncertainty in particle's position, i.e. (ΔX) . Then one can set $x = \epsilon \Delta X$. Therefore, equation (19) can be written as

$$\Delta A \geq 8\pi \Delta X \int_V T_{00} dV . \quad (20)$$

It is evident that the value of the integration on the right-hand side of equation (19) is exactly the energy of the outgoing particle. Since the integration is performed at a constant time over the whole space outside the black hole, it is legitimate to identify the energy of the particle as computed through the integration with the average energy of the particle given by equation (13).

$$\Delta A \geq 8\pi \epsilon \Delta X \Delta \omega , \quad (21)$$

Using the equations (15) and (18), we have that

$$\Delta A \geq 8\pi \epsilon \left(\frac{1}{2} + 2T_H^2 \right) . \quad (22)$$

In a way that

$$\Delta A_{min} = 8\pi \epsilon \left(\frac{1}{2} + 2T_H^2 \right) . \quad (23)$$

The equation (23) give us an area spectrum which is not equally spaced, but increases as the black hole evaporates, in a different way from the Bekenstein proposal.

5. Conclusions

The black hole area spectrum has been studied in the framework of tunneling mechanism, and in the presence of a Generalized Uncertainty Principle (GUP). We get the expression for the change in the black-hole area in this framework. The results implies in a non evenly spaced spectrum for the black hole area in a different way from the Bekenstein proposal. In this case, the lines of the area spectrum becomes more and more spaced as the black hole evaporates.

References

- [1] S.W. Hawking, Particle Creation by Black Holes, *Commun.Math.Phys.* **43** (1975) 199–220.
- [2] S.W. Hawking, Breakdown of Predictability in Gravitational Collapse, *Phys.Rev.D* **14** (1976) 2460–2473.
- [3] J.D. Bekenstein, Black holes and the second law, *Lett. Nuovo Cim.* **4** (1972) 737–740.
- [4] J.D. Bekenstein, *Lett. Nuovo Cimento* **11** (1974) 467.
- [5] S. Hod, *Phys. Rev. Lett.* **81**, (1998) 4293.
- [6] G. Kunstatter, *Phys. Rev. Lett.* **90**, (2003) 161301.
- [7] E. Berti, V. Cardoso and S. Yoshida, Highly damped quasinormal modes of Kerr black holes: A Complete numerical investigation,” *Phys. Rev. D* **69**, 124018 (2004).
- [8] U. Keshet and S. Hod, Analytic study of rotating black-hole quasinormal modes, *Phys. Rev. D* **76**, 061501 (2007) [arXiv:0705.1179 [gr-qc]].
- [9] U. Keshet and A. Neitzke, Asymptotic spectroscopy of rotating black holes, *Phys. Rev. D* **78**, 044006 (2008).
- [10] R. Banerjee, B. R. Majhi and E. C. Vagenas, A Note on the Lower Bound of Black Hole Area Change in Tunneling Formalism,” *Europhys. Lett.* **92**, 20001 (2010).
- [11] S. Iso, H. Umetsu and F. Wilczek, Anomalies, Hawking radiations and regularity in rotating black holes, *Phys. Rev. D* **74**, 044017 (2006).
- [12] K. Umetsu, Hawking Radiation from Kerr-Newman Black Hole and Tunneling Mechanism, *Int. J. Mod. Phys. A* **25**, 4123 (2010).
- [13] R. Banerjee and B. R. Majhi, “Connecting anomaly and tunneling methods for Hawking effect through chirality,” *Phys. Rev. D* **79**, 064024 (2009).
- [14] R. Banerjee and B. R. Majhi, Hawking black body spectrum from tunneling mechanism, *Phys. Lett. B* **675**, 243 (2009).
- [15] A.K. Raychaudhuri S. Banerji and R. Banerjee, *General Relativity, Astrophysics, and Cosmology* (Springer, New York).
- [16] F. Scardigli, Generalized uncertainty principle in quantum gravity from micro - black hole Gedanken experiment,” *Phys. Lett. B* **452**, 39 (1999)
- [17] J. D. Bekenstein, Generalized second law of thermodynamics in black hole physics, *Phys. Rev. D* **9**, 3292 (1974).